

POINCARÉ–BIRKHOFF–WITT THEOREM FOR QUADRATIC ALGEBRAS OF KOSZUL TYPE

A. Braverman and D. Gaitsgory

School of Mathematical Sciences
Tel Aviv University
Ramat Aviv, 69978, Israel

0. Introduction

0.1. Homogeneous quadratic algebras. Let V be a vector space over some field k and let $T(V) = \bigoplus T^i$ be its tensor algebra over k . Fix a subspace $R \subset T^2 = V \otimes V$, consider the two-sided ideal $J(R)$ in $T(V)$ generated by R and denote by $Q(V, R)$ the quotient algebra $T(V)/J(R)$. This is what is known as (a homogeneous) *quadratic algebra*.

0.2. Nonhomogeneous quadratic algebras. In a similar way we define *nonhomogeneous quadratic algebras*, the main objects of our study. They are *filtered* analogs of graded homogeneous quadratic algebras.

Consider the natural filtration $F^i(T) = \{\bigoplus T^j \mid j \leq i\}$ of $T(V)$. Fix a subspace $P \subset F^2(T) = k \oplus V \oplus (V \otimes V)$, consider the two-sided ideal $J(P)$ in $T(V)$ generated by P and denote by $Q(V, P)$ the quotient algebra $T(V)/J(P)$. The algebra $Q(V, P)$ will be called a *nonhomogeneous quadratic algebra*.

0.3. Let $U = Q(V, P)$ be a nonhomogeneous quadratic algebra. It inherits a filtration $U_0 \subset U_1 \subset \dots \subset U_n \dots$ from $T(V)$. We would like to describe the associated graded algebra $\text{gr } U$.

Consider the natural projection $p : F^2(T) \rightarrow V \otimes V$ on the homogeneous component, set $R = p(P)$ and consider the homogeneous quadratic algebra $A = Q(V, R)$. We have the natural epimorphism $p : A \rightarrow \text{gr } U$.

Definition. A nonhomogeneous quadratic algebra $U = Q(V, P)$, or, more precisely, the subspace P of $F^2(T)$, is of *Poincaré–Birkhoff–Witt (PBW) type* if the natural projection $p : A = Q(V, R) \rightarrow \text{gr } U$ is an isomorphism.

0.4. Lemma. Suppose that the subspace $P \subset F^2(T)$ is of PBW-type. Then it satisfies the following conditions :

- (I) $P \cap F^1(T) = 0$;
- (J) $(F^1(T) \cdot P \cdot F^1(T)) \cap F^2(T) = P$.

Proof is straightforward.

0.5. The main theorem. Let $U = Q(V, P)$ be a nonhomogeneous quadratic algebra. Take $R = p(P) \subset T^2(V)$ and consider the corresponding homogeneous quadratic algebra $A = Q(V, R)$.

Suppose A is a Koszul algebra (see 3.4). Then conditions (I) and (J) above imply that the subspace P and hence the algebra U , is of PBW-type.

The theorem will be proved in section 4 by means of deformation theory; in particular, it will provide a new proof of the classical PBW-theorem. Note, that a slightly weaker version of the above result (for $P \subset T^2(V) \oplus T^1(V)$) was proved by Polishchuk and Posetselsky ([PoP]) by methods different from ours.

0.6. Example. If $A = S(V)$ (the symmetric algebra of V) and P does not have the scalar component, then R is equal to $\Lambda^2(V)$, i.e., the space of all antisymmetric 2-tensors (more precisely, it is the space generated by all tensors of the form $x \otimes y - y \otimes x$; this is important when $\text{char } k = 2$). In this case P can be represented as a graph of some map $\alpha : \Lambda^2 V \rightarrow V$. The classical Poincaré–Birkhoff–Witt theorem asserts then that P is of PBW-type if and only if α satisfies the Jacobi identity. In §4 we will see that this is equivalent to condition (J) (which explains this abbreviation).

0.7. Contents. This paper is organized as follows: in §§1 and 2 we briefly explain the idea of our proof and review some basic facts concerning deformations and cohomology of associative algebras. In §3 we define Koszul algebras and reformulate the main problem in terms of certain explicit identities (which can be considered as generalised Jacobi identities) and in §4 we prove the above theorem. Since we use a somewhat nontraditional definition of Koszul algebras we prove its equivalence with the usual one in the appendix together with a review of some other basic properties of Koszul algebras.

0.8. Acknowledgments. It is a great pleasure for us to express our deep gratitude to Joseph Bernstein for posing the problem and valuable discussions. We are also grateful to Arkadii Vaintrob who introduced us to Koszul algebras.

1. PBW-theorem and Deformations

1.1. First, we will briefly explain the idea of the proof of Main Theorem. We start with the following simple observation. Suppose we are given a k -algebra U with an increasing filtration: $U = \bigcup_{i=0}^{\infty} U_i$ and let $\text{gr } U$ be its associated graded algebra: $\text{gr } U = \bigoplus_{i=0}^{\infty} U_i/U_{i-1}$. Then we can always construct a family of associative filtered algebras over the affine line $\text{Spec}(k[t])$ (this family is an algebra over the ring $k[t]$) such that *its fiber over point $t = 0$ is isomorphic to $\text{gr}U$ and its fiber over any point $t = \lambda \neq 0$ is isomorphic to U* . Namely, we define this algebra \mathcal{U} as follows:

$$\mathcal{U} = \{\sum u_i t^i \mid u_i \in U_i\} \subset U \otimes_k k[t] = U[t].$$

(The algebra \mathcal{U} is usually called the Rees ring of U). The verification of the property italicized is straightforward. One can immediately see that \mathcal{U} becomes a graded $k[t]$ -algebra if we set $\deg t = 1$. (It is easy to see that in the situation described above every

fiber over $t \neq 0$ has a natural structure of a filtered algebra and *not* of a graded one since $\deg t = 1$.)

The idea of the proof of the theorem is to reverse this process – namely, starting with a quadratic algebra $A = Q(V, R)$, where $R \subset T^2(V)$ and P are as in the introduction, to construct a family of filtered algebras over $\text{Spec}(k[t])$ such that its fiber at $t = 0$ is $Q(V, R)$, and its fiber at $t = 1$ is $Q(V, P)$ and such that the corresponding associated graded family is trivial. To do this rigorously we need the notion of a *graded deformation*.

1.2. Graded deformations. Let A be a (positively) graded associative algebra over a field k . By an *i-th level graded deformation* of A we will mean a graded $k[t]/k[t]t^{i+1}$ -algebra A_i , which is free as a $k[t]/k[t]t^{i+1}$ -module, together with an isomorphism of $A_i/tA_i \cong A$. (Here $\deg t = 1$.) By a *graded deformation* A_t of A we mean a graded algebra over the polynomial ring $k[t]$ (remember that $\deg t = 1$), which is free as a module over this ring, together with an isomorphism $A_t/tA_t \cong A$.

1.3. Let $\mathcal{E}(A)$ denote the category of all graded deformations of A where the morphisms are by definition isomorphisms of graded $k[t]$ -algebras (by definition this category is a groupoid). Analogously let $\mathcal{E}_i(A)$ denote the *groupoid of all i-th level graded deformations of A* . We denote by \mathcal{F}_i the functor from $\mathcal{E}_{i+1}(A)$ to $\mathcal{E}_i(A)$ given by reduction modulo t^i . The following lemma is straightforward.

Lemma. *Reductions modulo t^i define an equivalence between the category $\mathcal{E}(A)$ and the inverse limit of the categories $\mathcal{E}_i(A)$ with respect to the functors \mathcal{F}_i .*

1.4. Graded deformations have the following property which explains their importance for us: for every $\lambda \in k$ the fiber of \mathcal{A} at λ has a natural structure of a filtered algebra and its associated graded algebra $\text{gr}(A_\lambda)$ is canonically isomorphic to the original algebra A .

How to continue graded deformations from one level to another? The following proposition answers the question.

1.5. Proposition. a) *The set of isomorphism classes of objects of \mathcal{E}_1 canonically identifies with $H_{-1}^2(A, A)$.*

(Here $H^i(A, A)$ denotes the graded Hochschild cohomology of A with coefficients in A (see 2.1) and subscript denotes the natural grading induced on these cohomology groups by the grading of A .)

b) *Let A_i be an object of $\mathcal{E}_i(A)$. Then the obstruction for its continuation to the $(i+1)$ -st level lies in $H_{-i-1}^3(A, A)$.*

c) *Let A_i be as in (b). Then the set of isomorphism classes of continuations of A_i to the $(i+1)$ -st level has a natural structure of an $H_{-i-1}^2(A, A)$ -homogeneous space.*

Proof of this proposition will be sketched in the next section.

1.6. Remark. All definitions and statements of 1.2–1.5 are essentially contained in the classical work [G] of M. Gersthaber. Here we need a slightly different version of [G] relevant for the graded situation.

2. Cohomology and Deformations

Here we review some facts concerning cohomology of associative algebras and prove the proposition stated in the previous section.

2.1 Hochschild cohomology. Let A be any associative k -algebra and M any A -bimodule, i.e., left $A \otimes A^\circ$ -module (where A° is the same algebra A with multiplication $a \cdot b = ba$). We will be interested in $Ext_{A \otimes A^\circ}(A, M)$. This cohomology can be computed using the resolution $B^i(A)$ of A (*bar resolution*)

$$B^i(A) = A \otimes T^i(A) \otimes A \quad (T^i(A) = A^{\otimes i})$$

$$\partial^i(a_0 \otimes a_1 \cdots \otimes a_i \otimes a_{i+1}) = \sum_{k=0}^i (-1)^k (a_0 \otimes a_1 \otimes \cdots \otimes a_k a_{k+1} \otimes \cdots \otimes a_{i+1}) .$$

One can easily verify that it gives a projective (in fact, even free) resolution of A as an $A \otimes A^\circ$ -module ($B^i(A)$ has natural structure of a free left $A \otimes A^0$ -module). It is obvious that $\text{Hom}_{A \otimes A^\circ}(B^i(A), M)$ is the same as $\text{Hom}_k(T^i(A), M)$ for any A -bimodule M . Hence, we see that $Ext_{A \otimes A^\circ}(A, M)$ can be computed by means of the complex: $C^i(A, M) = \text{Hom}(T^i(A), M)$ with the differential

$$df(a_1, \dots, a_{i+1}) = a_1 f(a_2, \dots, a_{i+1}) + (-1)^{i+1} f(a_1, \dots, a_i) a_{i+1} +$$

$$\sum_{k=1}^i (-1)^k f(a_1, \dots, a_k a_{k+1}, \dots, a_{i+1}), \text{ for } f \in C^i(A, M).$$

This cohomology is called the *Hochschild cohomology* of A with coefficients in M . We denote it by $H^i(A, M)$.

2.2. Graded Hochschild cohomology. Let now A be a graded algebra and let M be a graded A -bimodule. In this case we shall slightly modify the above definition of $H^i(M)$. Namely, we set $H^i(M)$ to be equal to $Ext_{A \otimes A}(A, M)$ in the category of graded A -bimodules. This cohomology can be computed by means of the complex $C_{gr}^i(A, M)$ (subscript "gr" means "graded") where

$$C_{gr}^i = \bigoplus_{j=-\infty}^{\infty} \text{Hom}_j(T^i(A), M)$$

and $\text{Hom}_j(T^i(A), M)$ is the set of all homogeneous maps of degree of homogeneity j (the differential is defined as in 2.1). From now on (except for subsection 5.4) we shall deal only with graded algebras and modules, so, "Hochschild cohomology" will always mean "graded Hochschild cohomology".

2.3. Proof of Proposition 1.5(a). When we identify A_1 (our first level deformation) with $A \oplus A \cdot t$, the multiplication in A is described by a k -linear map $f: A \otimes A \rightarrow A$, namely

$$a \times b = ab + tf(a, b) \quad (\deg(f) = -1).$$

The associativity condition is rewritten in terms of f as:

$$f(a, b)c - f(ab, c) + f(a, bc) - af(b, c) = 0 \text{ for any } a, b, c \in A.$$

In other words, f must be a Hochschild 2-cocycle. It is easy to check that two infinitesimal deformations are isomorphic if and only if the corresponding cocycles are cohomologous.

2.4. Proof of Proposition 1.5(b). Suppose we have an n -th level deformation A_n . If we identify A_n with $A \otimes k[t]/k[t]t^{n+1}$, then the multiplication in A_n is described in terms of a sequence of maps $f_i: A \otimes A \rightarrow A$, $\deg(f_i) = -i$, $i = 1, \dots, n$ (the product of any two elements a and b of A in A_i is defined by

$$a \times b = ab + \sum_{i=1}^n f_i(a, b)t^i.$$

If f_{n+1} defines a continuation of A_n , then it satisfies:

$$\sum_{i=1}^n f_i(f_{n-i}(a, b), c) - f_i(a, f_{n-i}(b, c)) = f_{n+1}(a, bc) - f_{n+1}(ab, c) + f_{n+1}(a, b)c - af_{n+1}(b, c)$$

for any $a, b, c \in A$. In other words, f_{n+1} must have a prescribed coboundary, namely the left hand side of the above formula, whose image in cohomology is precisely the obstruction to the existence of a continuation from the i -th level to the next one (it is not difficult to show that the above expression is always a cocycle).

2.5. Proof of Proposition 1.5(c). As in (b) we see, that we may vary f_{n+1} by a cocycle. It is also easy to verify that two continuations are isomorphic if the corresponding f_{n+1} 's are cohomologous. This means that the second cohomology group acts transitively on the set of isomorphism classes of continuations.

3. Koszul Algebras and Certain Identities

3.1. Now let us return to quadratic algebras. Let V be a vector space over k and let $P \subset V$ satisfy the conditions (I) and (J) of 0.2. Let R be the projection of P to the second homogeneous component. Now we ask: is the natural surjection from $Q(V, R)$ to $\text{gr}(Q(V, P))$ an isomorphism? In other words, under what conditions is P of PBW type?

In this section we shall reformulate this problem and define the class of algebras (Koszul algebras) for which we can completely answer the above question.

3.2. As it was said in the introduction, conditions (I) and (J) are necessary for P to be of PBW-type. In fact, they only require the injectivity of the map $p: Q(V, R) \rightarrow \text{gr}Q(V, P)$ on the 3-rd term of the natural increasing filtration arising from the grading. From now on we will assume that all nonhomogeneous quadratic algebras we consider satisfy (I). Then the subspace $P \subset F^2(V)$ can be described in terms of two maps $\alpha: R \rightarrow V$ and $\beta: R \rightarrow k$ as $P = \{x - \alpha(x) - \beta(x) \mid x \in R\}$.

3.3. Lemma. *The condition (J) can be rewritten in terms of α and β in the following way:*

- (i) $\text{Im}(\alpha \otimes \text{id} - \text{id} \otimes \alpha) \subset R$ (this map is defined on $(R \otimes V) \cap (V \otimes R)$);
- (ii) $\alpha \circ (\alpha \otimes \text{id} - \text{id} \otimes \alpha) = -(\beta \otimes \text{id} - \text{id} \otimes \beta)$;
- (iii) $\beta \circ (\text{id} \otimes \alpha - \alpha \otimes \text{id}) \equiv 0$.

Proof: Let $x \in (R \otimes V \cap V \otimes R)$. Then by definition of α and β we have:

$$\begin{aligned} \alpha \otimes \text{id}(x) + \beta \otimes \text{id}(x) - x &\in P \cdot T_1; \\ \text{id} \otimes \alpha(x) + \text{id} \otimes \beta(x) - x &\in T_1 \cdot P. \end{aligned}$$

Thus by (J) we have

$$\alpha \otimes id(x) - id \otimes \alpha(x) + \beta \otimes id(x) - id \otimes \beta(x) \in P.$$

This implies that

$$\begin{aligned} \alpha \otimes id(x) - id(x) \otimes \alpha(x) &\in R; \\ \alpha(\alpha \otimes id(x) - id \otimes \alpha(x)) &= -(\beta \otimes id(x) - id \otimes \beta(x)); \\ \beta(\alpha \otimes id(x) - id \otimes \alpha(x)) &= 0 \quad \square \end{aligned}$$

Our main theorem asserts, that conditions (i), (ii) and (iii) above enable us to build a graded deformation of $Q(V, R)$ such that its fiber at the point $t = 1$ is canonically isomorphic, as a filtered algebra, to $Q(V, P)$, provided $Q(V, R)$ is a Koszul algebra. In view of Remark 1.4 this means that our $Q(V, P)$ is of PBW type.

3.4. Definition. Let $A = Q(V, R)$ be a quadratic algebra. We say that it is of *Koszul type* if $H_j^i(A, M) = 0$ for all $i < -j$ and for every \mathbb{Z}^+ -graded A -bimodule M .

3.5. Example. Let V be a vector space. Denote by $S(V)$ its symmetric algebra with its usual grading. Then it is well known that $H^i(S(V), M)$ is a subquotient of $M \otimes \Lambda^i(V^*)$, where the elements of $\Lambda^i(V^*)$ have degree $-i$. Thus, one can see that $H_j^i(S(V), M)$ vanishes for $i < -j$, so the symmetric algebra is a Koszul algebra.

3.6. Koszul complex. There are several definitions of Koszul algebras. Some of them and a proof of their equivalence are discussed in the appendix. Here we shall also need the following one.

Let $A = Q(V, R)$. Define the following subcomplex $\tilde{K}^\cdot(A)$ of $B^\cdot(A)$. Set

$$K^i(A) = \bigcap_{j=0}^{i-2} V^{\otimes j} \otimes R \otimes V^{\otimes i-j-2}; \quad \tilde{K}^i(A) = A \otimes K^i \otimes A.$$

We have a natural imbedding of $K^i(A)$ into $T^i(V)$; hence, into $T^i(A)$. Thus, we can regard \tilde{K}^i as an $A \otimes A^\circ$ -submodule of $B^i(A)$. One can verify by a direct computation that $\tilde{K}^\cdot(A)$ forms a subcomplex of $B^\cdot(A)$. It is called the *Koszul complex associated to A* . This complex is not always acyclic. However, the following is true:

Claim. A is a Koszul algebra if and only if

$$H^i(\tilde{K}^\cdot(A)) = \begin{cases} 0 & \text{if } i > 0 \\ A & \text{if } i = 0 \end{cases}$$

In other words, A is Koszul iff the map of complexes $\tilde{K}^\cdot(A) \rightarrow B^\cdot(A)$ is a quasiisomorphism.

This claim will be proven in the appendix.

3.7. Deformations of Koszul algebras. In the proof of the main theorem we shall need the following statement about functors $\mathcal{F}_i(A)$ for a Koszul algebra A .

Proposition. *Let A be a Koszul algebra. Then*

(i) the functors $\mathcal{F}_i(A) : \mathcal{E}_i(A) \rightarrow \mathcal{E}_{i-1}(A)$ are monomorphic on isomorphism classes of objects for $i > 2$

(ii) the functors $\mathcal{F}_i(A)$ are epimorphic on isomorphism classes of objects for $i > 3$

Proof: It follows from the definition of Koszul algebras that $H_{-i-1}^2(A, A)$ vanishes for $i > 2$. Hence Proposition 1.5(c) implies (i). Analogously, $H_{-i-1}^3(A, A)$ vanishes for $i > 3$, hence Proposition 1.5(b) implies (ii). \square

3.8. Remark. In fact one can prove that under the conditions of 3.7. the functors $\mathcal{F}_i(A)$ are also surjective on morphisms for $i > 3$ and hence, the categories $\mathcal{E}(A)$ and $\mathcal{E}_i(A)$ have the same isomorphisms classes of objects. The reason for that is again cohomologous and it will not be explained in this paper.

4. The Main Theorem

4.1. Theorem. *Let $A = Q(V, P)$ be a Koszul algebra and let $\alpha : R \rightarrow V$ and $\beta : R \rightarrow k$ be linear maps that satisfy identities (i), (ii) and (iii) of 3.3. Then there exists a graded deformation of A such that its fiber at the point $t = 1$ is canonically isomorphic to $U = Q(V, P)$ defined by means of α and β .*

4.2. Plan of the proof. First we show in subsection 4.3 that a map α satisfying 3.3(i) defines a first level graded deformation of A uniquely up to an isomorphism. Next we show that a map β , satisfying 3.3(ii) defines a continuation of this deformation to the second level (again, uniquely up to an isomorphism). Then condition 3.3(iii) ensures that this deformation lies in the image of the functor \mathcal{F}_3 , i.e. it can be continued to the third level. Thus Proposition 3.7 implies that it extends to some graded deformation of A , which in fact in view of Remark 3.8 will be uniquely defined. Finally we prove that this deformation satisfies the conditions of 4.1.

4.3. Extension to the first level. Both α and β are defined on R which is equal to K^2 . Obviously, we may regard them as $A \otimes A^\circ$ -linear maps from \tilde{K}^2 to A . We denote these maps also by α and β . Then we see that condition (i) means that $d\alpha = 0$ (i.e. α is a 2-cocycle in the complex $\text{Hom}_{A \otimes A^\circ}(\tilde{K}^\cdot, A)$). Thus, it defines a cohomology class in $H^2(A, A)$.

Let us prove that we can find a Hochschild 2-cocycle f_1 such that $f_1|_{K^2} = \alpha$ and f_1 is homogeneous (of degree -1). Let \tilde{f}_1 be any Hochschild 2-cocycle which defines the same cohomology class as α . Then $\tilde{f}_1 - \alpha|_{K^2} = d\omega$ for some Koszul 1-cochain ω of degree -1 . Next, we can find a Hochschild 1-cochain ω' of degree -1 such that $\omega'|_{K^1} = \omega$. Define f_1 to be $\tilde{f}_1 - d\omega'$. We set this f_1 to be the first term of the deformation.

4.4. Extension to the second level. Now, by (ii) $d\beta = \alpha \circ (\alpha \otimes id - id \otimes \alpha)$ in the Koszul complex. Let us prove that there exists a Hochschild 2-cochain f_2 of degree -2 satisfying

$$df_2 = f_1 \circ (f_1 \otimes id - id \otimes f_1) \quad \text{and} \quad f_2|_{K^2} = \beta.$$

We can find a Hochschild 2-cochain \tilde{f}_2 such that $\tilde{f}_2|_{K^2} = \beta$. Consider the cocycle $\gamma = d\tilde{f}_2 - f_1 \circ (f_1 \otimes id - id \otimes f_1)$. Then it follows from (ii) that γ is cohomologically trivial since

$\gamma|_{K^3} = 0$. Hence, there exists a Hochschild 2-cochain μ of degree -2 such that $d\mu = \gamma$. Then $\mu|_{K^2}$ is a cocycle and one can find (as in 4.2) a Hochschild 2-cocycle μ' of degree -2 whose restriction to K^2 is equal to μ . Now it is easy to see that if we define f_2 to be equal to $\tilde{f}_2 - \mu + \mu'$, then it will satisfy the above identity.

4.5. Extension to the third level. Let f_2 be the second term of the deformation. The second obstruction is given by the cocycle

$$\delta = f_1 \circ (f_2 \otimes id - id \otimes f_2) - f_2 \circ (f_1 \otimes id - id \otimes f_1).$$

One can easily verify that (iii) is equivalent to the vanishing of δ in $H^3(A, A)$. Hence, we can continue our deformation to the third level.

4.6. Now we see that Proposition 3.7 implies that our deformation extends to some (unique) graded deformation of A .

Let us now prove that the fiber \tilde{U} of this deformation at $t = 1$ is isomorphic to U . We have a map of vector spaces $\tilde{\varphi}: V \rightarrow \tilde{U}$, which by the conditions imposed on f_1 and f_2 gives rise to a map $\varphi: U \rightarrow \tilde{U}$, since the multiplication in A_t is represented as

$$a \times b = ab + \sum_{i=1}^{\infty} f_i(a, b)t^i.$$

We know that there is a canonical isomorphism between A and $\text{gr}(\tilde{U})$. Obviously, the composition map

$$A \xrightarrow{p} \text{gr}U \xrightarrow{\text{gr}\varphi} \text{gr}\tilde{U} \rightarrow A$$

is id_A . Thus p , being a surjection, is an isomorphism and φ is an isomorphism, too. \square

4.7. Remark. In fact one can prove that the deformation that we have constructed is actually isomorphic to the Rees ring of U (see 1.1.)

5. Examples

5.1. The classical PBW-theorem. Let $A = S(V)$. Then $R = \Lambda^2(V)$. Consider the case $\beta \equiv 0$. Then a straightforward computation shows that heading (ii) of Lemma 3.3 is the Jacobi identity for α . Thus, V with the bracket α becomes a Lie algebra and so 4.1 gives us another proof of the classical PBW theorem.

5.2. The supercase. Let $V = V_0 \oplus V_1$ be a superspace and $A := S(V_0) \otimes \Lambda(V_1)$ be its symmetric algebra. Suppose as in 5.1 that $\beta \equiv 0$. Then condition (ii) is the super-Jacobi identity for α . Thus, we see that α defines a Lie superalgebra structure on V (note that conditions (i) and (ii) provide us with a definition of a Lie superalgebra over a field of any characteristic); hence, 4.1 proves also the superversion of the PBW theorem (the fact that A is a Koszul algebra follows, for example, from Appendix).

5.3. Weyl and Clifford algebras. Let A be either $S(V)$ or $\Lambda(V)$ and let α be identically zero. Then the only restriction on β is $\beta \otimes id - id \otimes \beta = 0$ that any map $\beta : R \rightarrow k$ clearly satisfies (in our case R is either $S^2(V)$ or $\Lambda^2(V)$). Hence, Theorem 4.1 proves also the PBW theorem for Weyl and Clifford algebras.

5.4. The standard complex. We conclude this section with a generalization of the standard complex for Lie algebras to quadratic algebras of PBW-type. Let $U = Q(V, P)$ be of PBW type and the corresponding homogeneous quadratic algebra be Koszul. Suppose that $P \subset T^2 \oplus T^1$. Then there exists a Koszul complex $\tilde{K}^\bullet(U)$ which gives a free resolution of U in the category of U -bimodules. Let us define \tilde{K}^i to be $U \otimes K^i \otimes U$, where the K^i are as in 3.6 for the corresponding homogeneous quadratic algebra. The condition $(\alpha \otimes id - id \otimes \alpha) = 0$ implies that the differential of the bar-resolution of U maps \tilde{K}^i into \tilde{K}^{i-1} .

To prove that the differential is exact, it suffices to notice that $\text{gr}(\tilde{K}^\bullet)$ is canonically isomorphic to the Koszul complex of the corresponding homogeneous quadratic algebra. Since passage to the associated graded module as a functor is faithful, the assertion follows.

As a corollary of this fact, we obtain, by tensoring with k , a free resolution of k in the category of left U -modules. For example, in the case when $U = U(\mathfrak{g})$ (the universal enveloping algebra of \mathfrak{g} , where \mathfrak{g} is a Lie algebra, we obtain the standard complex of \mathfrak{g} .

Appendix

Here we present a brief review of some basic properties of Koszul algebras.

A.1. Let A be a homogeneous quadratic algebra over a field k . We start with the following (traditional) definition of Koszul algebras. (In what follows we will prove that it is equivalent to the definitions given in 3.4 and 3.6.) Let us view k as a left A -module.

Definition. A quadratic algebra A is called *Koszul* if $\text{Ext}_j^i(k, k) = 0$ for $i \neq -j$ (where $\text{Ext}^i(k, k)$ is taken in the category of left A -modules).

A.2. Proposition. Let $\tilde{K}^\bullet(A)$ denote the Koszul complex introduced in §3. The following conditions are equivalent:

- a) A is a Koszul algebra (in the sense of A.1).
- b) \tilde{K}^\bullet is a resolution of A in the category of A -bimodules.
- c) $\tilde{K}^\bullet \otimes_A k$ is a resolution of k in the category of left A -modules.
- d) $H_j^i(A, M)$, where A is considered as an A -bimodule, vanishes for any \mathbb{Z}^+ -graded A -bimodule M and $i < -j$.
- e) $\text{Ext}_j^i(k, N)$, where k is considered as a left A -module, vanishes for any \mathbb{Z}^+ -graded A -module N and $i < -j$.

Remark. If one of the equivalent conditions of A.2 is satisfied, then $\tilde{K}^\bullet \otimes_A k$ is the Koszul resolution of k in the sense of [Pr].

A.3. In the proof of the above proposition we will need the following general result.

Lemma. . Let $P^\bullet = \bigoplus_{i=-\infty}^{\infty} P_i^\bullet$ be a graded complex of graded A -modules over a \mathbb{Z}^+ -graded k -algebra A with $A_0 = k$. Suppose that these graded A -modules are free with homogeneous generators and with homogeneous of degree 0 differentials and such that $P_i^\bullet = 0$ for $i \ll 0$. Suppose that $\tilde{P}^\bullet = P^\bullet \otimes_A k$ is exact. Then P^\bullet is exact.

Proof. Let us denote by P_i^\bullet the subcomplex of P^\bullet , whose chain groups P_i^j are the subgroups of P^j generated by elements of degree $\leq i$. Then P^\bullet is the direct limit of P_i^\bullet , so it will be enough to prove that if $\tilde{P}_i^\bullet = P_i^\bullet \otimes_A k$ are exact, then all P_i^\bullet are exact because they P_i^\bullet is a direct summand of $P^\bullet \otimes_A k$. So, we are reduced to the case when the chain groups in P^\bullet are generated by elements of degree $\leq i$.

Let us fix this i and proceed by the descending induction on j : j is the minimal m for which P^\bullet is nontrivial in degree m . (This j is well-defined since P_i^\bullet is assumed to be zero for $i \ll 0$.)

For $j = i + 1$ the assertion is trivially satisfied, so we have to carry out the inductive step.

Let the assertion be true for $j + 1$; let us prove that it is true for j . Let P_j^\bullet be as before. Then P^\bullet / P_j^\bullet satisfies the inductive hypothesis, and P_j^\bullet is exact since it is equal to $\tilde{P}_j^\bullet \otimes_k A$, by the minimality of j . Hence, the assertion follows. \square

A.4. Proof of Proposition A.2. (e) \Rightarrow (d): follows from the fact that $H_j^i(A, M)$ (in the category of A -bimodules) is naturally isomorphic to $Ext_j^i(k, M)$ (in the category of left A -modules) for every bimodule M .

Implications (b) \Rightarrow (d), (c) \Rightarrow (e) follow immediately from exactness of the Koszul complex.

(b) \Rightarrow (c) is easy, since we can tensor the Koszul resolution of A as an A -bimodule by k and obtain the Koszul resolution of k , as a left A -module.

The implication (c) \Rightarrow (b) is a direct corollary of Lemma A.3.

Implication (c) \Rightarrow (a) is a consequence of the exactness of the Koszul complex.

(a) \Rightarrow (e): Let P^\bullet be a minimal free graded resolution of k . Then one has $Ext^i(k, k) = \text{Hom}_A(P^i, k)$. In particular, $Ext_j^i(k, k) = 0$ for $i < -j$ imply that P^i is trivial in degrees $i < j$, so $Ext_j^i(k, N)$ vanishes for every module N when $i < -j$.

In order to prove (e) \Rightarrow (c) we shall proceed by induction: let

$$A \otimes K^i \longrightarrow A \otimes K^{i-1} \longrightarrow \dots \longrightarrow A \otimes V \longrightarrow k$$

be exact. Then we see that $Ext_j^{i+1}(k, N) = 0$ for any N and also $j > i + 1$ implies that $\ker \left(A \otimes \tilde{K}^i \longrightarrow A \otimes \tilde{K}^{i-1} \right)$ is generated by the elements of degree $\leq i + 1$, i.e., these generators lie in $V \otimes K^i \subset V^{\otimes i+1}$.

At the same time, they have to lie in $R \otimes V^{\otimes i-1}$. In other words, the generators of this kernel lie in $R \otimes K^{i-2} \cap V \otimes K^{i-1}$, so $A \otimes K^{i+1}$ is surjectively mapped onto the kernel. This finishes the proof of Proposition. \square

A.5. The dual (co)algebra. We conclude this appendix with a description of some properties of *the dual algebra of a quadratic algebra* (see [Ma]). Let us start with a homogeneous quadratic algebra $Q(V, R)$. Then we may consider a graded subspace of $T(V)$, namely $\oplus_{i=0}^{\infty} K^i$. This subspace admits a natural coalgebra structure, induced from that of $T(V)$.

Suppose now that V is finite dimensional. Then we may also consider a dual object to this coalgebra, namely $\oplus_{i=0}^{\infty} K^{i*}$. It has a natural algebra structure and it is called the *dual quadratic algebra to A* and denoted by $A^!$.

A.6. Proposition. *A is a Koszul algebra if and only if $A^!$ is; in this case $Ext_j^i(k, k) = \begin{cases} 0 & \text{for } i \neq -j \\ A_i^! & \text{for } i = j \end{cases}$.*

Proof. It clearly suffices to prove the statement in one direction, since $A = (A^!)^!$. Consider the Koszul complex for A in degree i :

$$(A_i^!)^* \longrightarrow \cdots \longrightarrow (A_{i-k}^!)^* \otimes A_k \longrightarrow \cdots \longrightarrow A_i.$$

It is exact by the assumption. When we dualize it with respect to k , we obtain the Koszul complex for $A^!$ in degree i . \square

References

- [G] M.Gerstenhaber, Deformations of rings and algebras, *Ann. of Math.(2)*, **79**, 1964, 59-103
- [GeS] M.Gerstenhaber and S.D.Schack, Algebraic cohomology and deformation theory, in: "Deformation theory of algebras and structures and applications", M.Hazevinkel and M.Gerstenhaber eds., Kluwer, Dordrecht, 1988, 11-264.
- [Ma] Y.I.Manin, Quantum groups and noncommutative geometry, Montreal University, 1988.
- [PoP] A.E.Polishchuk and L.E.Positselsky, On quadratic algebras (in Russian), preprint, Moscow University, 1990.
- [Pr] S.Priddy, Koszul resolutions, *Trans. AMS*, **152**, 1970, 39-60